A comparative study of membrane finite elements based on the strain approach

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ABSTRACT. This research work deals with the extension two rectangular membrane finite elements to have the quadrilateral shape in order to extend the use of these elements for more general membrane structures. The strain approach has been used for the development of the displacement field of these two elements which have been implemented in the ABAQUS code. One of these elements has only two translations (u and v) per node (SBRIE) whereas the other element contains in addition an in-plane rotation per node at each of the four corner nodes (SBQE). Numerical evaluation has been conducted through a series of tests related to linear plane structures and the accuracy and robustness of the strain approach have been demonstrated compared to theoretical and other finite elements.

KEYWORDS: *finite element, linear analysis, strain approach, membrane, drilling, quadrilateral*

1. Introduction

The development of finite element models for linear and non-linear analysis in structural engineering and mechanics is a mature subject considering different formulations.

Finite element modeling of structures requires the use of a large number of discretization elements to achieve a certain accuracy of the analysis [HIM 15].

The formulation of finite elements based on the strain approach has continued in recent years and many elements were developed. Many researchers continue to be preoccupied with the problem of the formulation of new elements, and further development of improved algorithms for special phenomena [HAM 11].

This approach is based on the calculation of the exact terms representing all the rigid body modes and the other components of the displacement functions; which are based on assumed independent strain functions in so far as it is allowed by the compatibility equations. Direct interpolation based on the strain approach provides a better precision on these values and on constraints and displacements (obtained by integration); compared to the classic formulation where deformations are obtained by derivation of the chosen displacement fields. The main advantages of this approach are:

1) Easy satisfaction of the main two convergence criteria bound directly to strains (constant strains and rigid body movement).

2) Effortlessly decoupling of the various strain components (a field of uncoupled displacements generates coupled strains).

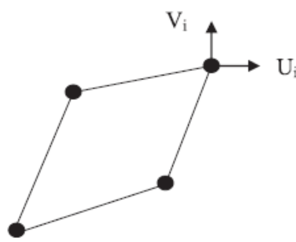
3) Possibility of enriching the field of displacements by terms of high order without the introduction of intermediate nodes or of supplementary degrees of freedom (allowing so to treat the problem of locking) [HAM 14].

In this paper the strain approach has been used for the development of the displacement field of two elements which have been implemented into an interface element in the UEL user subroutine in the finite element code ABAQUS. One of these elements has only two translations (u and v) per node whereas the other element contains in addition an in-plane rotation per node at each of the four corner nodes.

2. Description of strain based elements used

- ***SBRIE: Strain based rectangular in-plane elements with four nodes [SAB 95]***

This quadrilateral membrane finite element based on the strain approach has only two translations (u and v) per node for the four nodes. The displacement field is:

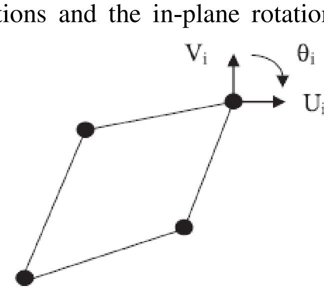


$$U = a_1 - a_3 y + a_4 x + a_5 xy - a_7 \frac{y^2}{2} + a_8 \frac{y}{2}$$

$$V = a_2 + a_3 y - a_5 \frac{x^2}{2} + a_6 y + a_7 xy + a_8 \frac{x}{2}$$

- ***SBQE: Strain based rectangular [REB 14]***

This quadrilateral membrane finite element with drilling rotation based on the strain approach has the three degrees of freedom at each of the four nodes (the two translations and the in-plane rotation) and the displacement functions satisfy. The displacements field is:



$$U = a_1 - a_3 y + a_4 x + a_5 (x + y^2) + a_6 xy + 0.5 a_7 x^2 + 0.5 a_8 y^2$$
$$+ 0.5 a_9 y^2 + a_{10} xy^2 + a_{11} x^2 y^3$$

$$V = a_2 + a_3 x + a_5 y + 0.5 a_6 x^2 + a_7 (y + x^2) + a_8 xy + 0.5 a_9 y^2$$
$$- a_{10} x^2 y - a_{11} x^3 y^2 + a_{12} x^2$$

$$\theta = a_3 - 2a_5 y + a_7 x - a_9 y - 2a_{10} xy - 3a_{11} x^2 y^2 + a_{12} x$$

The nodal displacements vector and the strain functions can be written respectively as:

$$[U] = [C][A]$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [Q][A]$$

The stiffness matrix $[K_e]$ can be calculated from the well known expression:

$$[K^e] = [C]^{-T}\left(\iint [Q]^T[D][Q]\,dxdy\right)[C]^{-1} = [C]^{-T}[K_0][C]^{-1}$$

Where [D] is the elasticity matrix, is the constant parameters vector $\{a_i\} = 1, \dots, 12$ and [Q] is the strain matrix.

Thus the matrix $[K_0]$ is numerically evaluated, and since the matrix [C] is not singular, its inverse can be also numerically evaluated and the element stiffness matrix $[K_e]$ can be obtained.

3. Numerical Tests

In order to validate the implementation in the ABAQUS code, various several examples for plane elasticity problems are presented. The elements used in this study are given:

- Q4: quadrilateral element with four nodes.
- SBRIE: Strain based rectangular in-plane elements with four nodes [SAB 95].
- SBQE: Strain based rectangular with three degrees of freedom (Ui, Vi, and in plane rotation hi) at each of the four nodes [REB 14].
- CPS4-c: quadrilateral element with four nodes of ABAQUS.
- CPS4-inc: incompatible quadrilateral element with four nodes of ABAQUS.

2.1. Mac-Neal's elongated cantilever beam *[REB 13]*

The problem of a cantilever beam shown in Fig.1 has been treated by Mac-Neal and Harder. The beam is subjected to a concentrated shearing force at the free end (P=1) and to a pure bending moment (M=10). It has Young's modulus E=10E7, Poisson's ratio v=0.3, and a thickness t=0.1. The results of the normalized deflection at the free end presented in Tables 1 and 2 show that:

- The SBQE gives better results than the elements, CPS4-c, SBRIE [SAB 95], and Q4 and it is in good agreement with the CPS4-inc element.
- The SBQE element has achieved excellent convergence to the reference solution in both problems (force shearing and pure bending).

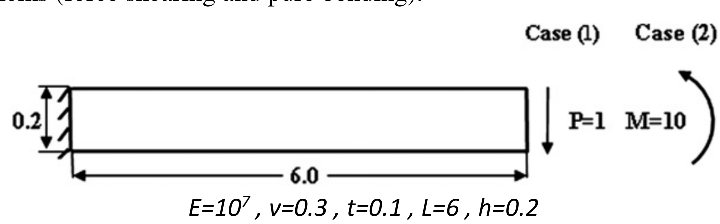


Figure 1. *Cantiliver beam subject at a straight shear or pure bending in the end*

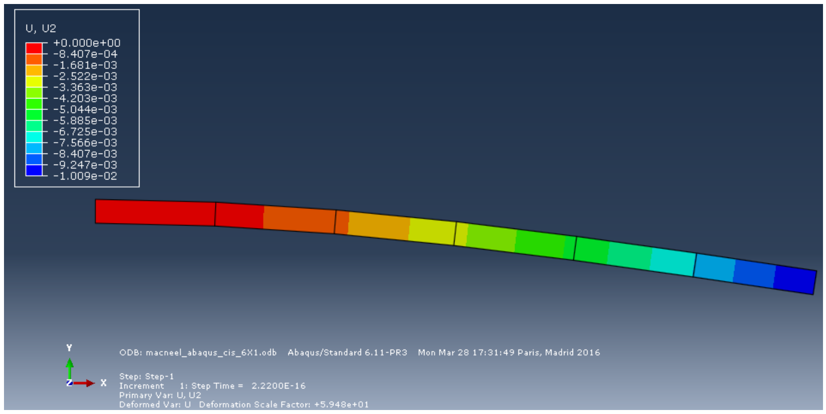


Figure 2. *Cantiliver beam of the calculation code ABAQUS*

Table 1. *Normalized deflection for Mac-Neal's elongated beam subjected to end shear*

Load case(1): Force shearing at the free end P=1					
Mesh	**CPS4-c**	**CPS4-inc**	**Q4**	**SBRIE**	**SBQE**
6x1	0.093	0.992	0.093	0.9036	0.992
12x1	0.093	0.9935	0.093	0.904	0.981
Reference solution : 1.000 (0.1080)					

Table 2. *Normalized deflection for Mac-Neal's elongated beam subjected to end pure bending*

Load case(2):Pure bending moment M=10					
Mesh	**CPS4-c**	**CPS4-inc**	**Q4**	**SBRIE**	**SBQE**
6x1	0.093	1.000	0.093	0.91	0.998
12x1	0.285	1.000	0.285	0.91	0.977
Reference solution : 1.000 (0.270)					

2.2. Mesh distortion test for beam bending *[YEO 06]*

The cantilever beam problem with two plane stress elements was suggested by Piltner et al. as a test for robustness of quadrilateral element to mesh distortion. In this test a beam under bending is analyzed with only two elements (Fig 3). The degree of distortion of the element is measured by the distortion parameter delta varies from 0 to 4. The detailed dimensions and properties are described in Fig.3 and the results are given in Table 3; the exact solution for this problem is v = 100.0. As shown in Table 4, the SBQE element is more accurate and robust than other elements.

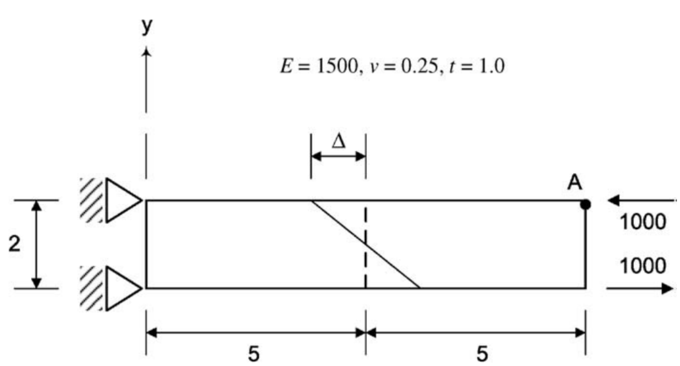


Figure 3. *The cantilever beam with Mesh distortion*

Table 3. *Displacement v of cantilever beam for different values of the mesh distortion parameter (delta)*

DELTA	CP4-c	CP4-inc	Q4	SBRIE	SBQE	EXACT
0	/	100	28,04	93,75	99,94	100
0,5	/	60,69	25,94	93,75	100	
1	/	35,47	21,48	93,76	99,92	
2	/	26,50	14.74	93,91	99,79	
3	/	27,77	11,9	94,22	99,79	
4	/	29.92	10,78	94,6	99,81	

2.3. Short cantilever beam of Allman *[REB 13]*

The short cantilever beam is subjected to uniform vertical load (with resultant W) as shown in Fig.4, and it is modeled by four rectangular elements. The results of the displacement at the free end presented in Table4. The SBQE and CPS4-inc converge better than that of Allman [ALL 88] towards the reference solution [TIM 51] for the normalized vertical displacement at point A.

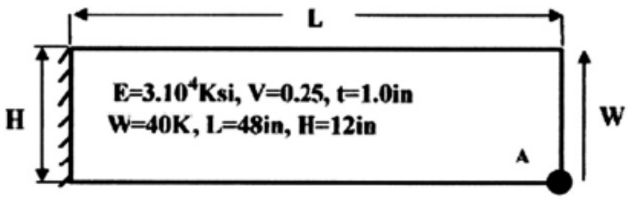


Figure 4. *Cantilever beam under a tip load*

Table 4. *Vertical displacement at point A*

Mesh	CPS4-c	CPS4-inc	Q4	SBRIE	SBQE	ALLMAN [ALL 88]
4x1	0.6822	0.9831	0.6822	0.924	1.012	0,852
Reference solution : 1.000 (0.3553)						

2.4. Cook's membrane problem *[YEO 06]*

The accuracy of the elements was investigated through Cook's membrane problem. Detailed dimensions and properties are given in Fig. 5 and the reference solution is: v =23.9 and the results of the displacement at the free end presented in Table 5. The results show that SBQE and CPS4-inc elements have good accuracy regardless of mesh density.

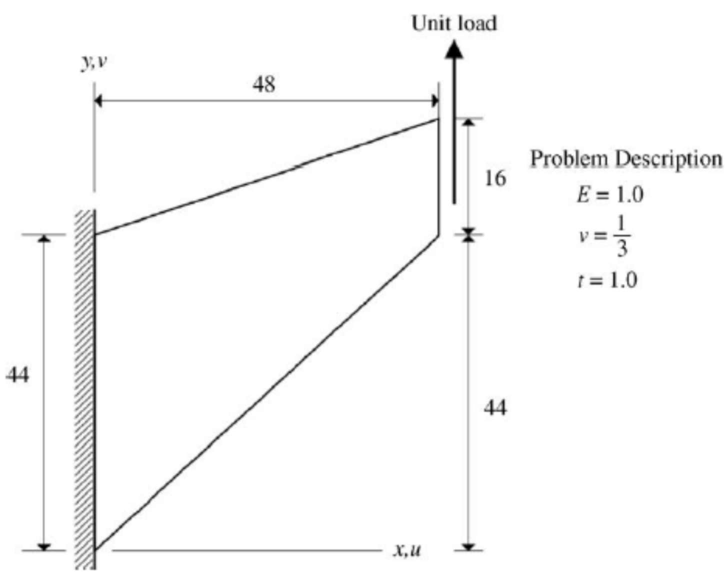


Figure 5. *Cook's membrane problem*

Table 5. *Tip displacements of Cook's problem for quadrilateral elements according to the mesh density*

Mesh	CPS4-c	CPS4-inc	Q4	SBRIE	SBQE
2x2	0.51	0.901	0.589	0.76	0.747
4x4	0.788	1.00	0.84	0.93	0.979
Reference solution : 1.000 (23.9)					

2.5. Shear wall with opening *[PAK 07]*

In order to show the efficiency, suitability, accuracy and superiority of the strain based elements, an attempt has been made to analyze shear wall structures with openings. Results of commercial packages, namely SAP-2000, STAADPRO given by [PAK 07] were used for comparisons. Fig. 6 shows geometry and material property of an eight story coupled shear wall.

The lateral displacement of the model at story 2, 4, 6 and 8 for all the FE codes and elements has been tabulated in Table 6. The CPS4-c, Q4 and SBQE elements and commercial software SAP-2000, STAADPRO reflect a comparable result. The SBRIE and CPS4-inc elements converged to greater deflection which are very similar by application of the OPT element.

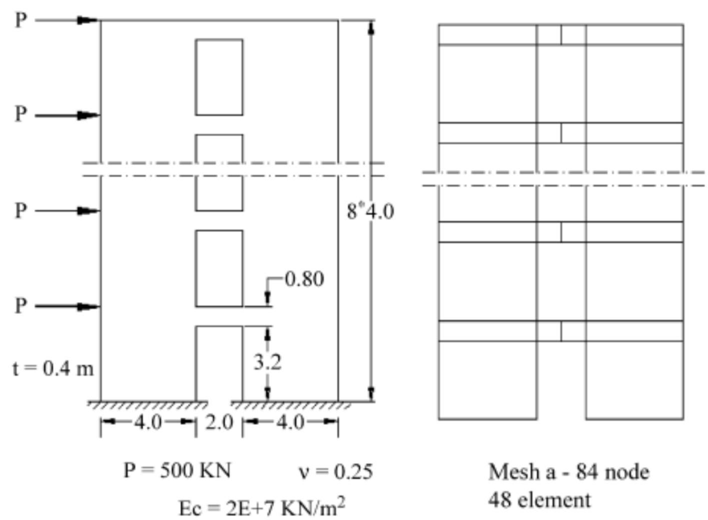


Figure 6. *Geometry and material of coupled shear wall*

Table 6. *Comparison of the lateral deflection at different story level*

Floor	SAP 2000 [PAK 07]	STAAD-PRO [PAK 07]	CPS4-c	CPS4-inc	Q4	SBRIE	SBQE	OPT element [PAK 07]
2	0.55	0.68	0.607	0.774	0.564	0.661	0.657	0.71
4	1.48	1.68	1.647	2.058	1.615	1.919	1.691	1.91
6	2.54	2.78	2.780	3.402	2.760	3.253	2.808	3.19
8	3.62	3.86	3.875	4.700	3.845	4.473	3.876	4.43

4. Conclusion

In this work, we have implemented all the elements used for different problems (Q4, SBRIE and SBQE) within the finite element software ABAQUS. The implementation was based on the user element subroutine UEL and enables the modeling of different problems using a single mesh that is easily generated. The Numerical results show that the strain based elements exhibits an excellent behavior for both regular and distorted mesh and could offer a significant efficiency and accuracy in problems analysis.

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